Available at
www.ElsevierMathematics.com
POWERED BY SCIENCE DIRECT
Discrete Applied Mathematics 135 (2004) 31–39
DISCRETE APPLIED MATHEMATICS
www.elsevier.com/locate/dam

Height of minor faces in plane normal maps ☆

O.V. Borodin[a,*,1], D.V. Loparev[b]

[a] *Sobolev Institute of Mathematics, Russian Academy of Sciences, Novosibirsk 630090, Russia*
[b] *Novosibirsk State University, Novosibirsk 630090, Russia*

Abstract

The height of a face in a plane graph is the greatest degree of a vertex incident with this face. Under appropriate necessary conditions we prove that any plane normal map has a 3-face of height at most 20, or a 4-face of height at most 11, or else a 5-face of height at most 5. The bounds 20 and 5 are shown to be the best possible. © 2002 Elsevier B.V. All rights reserved.

Keywords: Graph; Plane graph; Structure; Weight

1. Introduction

A plane normal map is a plane graph in which the degree $d(v)$ of each vertex v and the size $r(f)$ of each face f is at least three.

For a face f consider a vector $D(f)=(d_1,\ldots,d_r)$, where $r=r(f)$ and d_i's are the degrees of vertices incident with f in the non-decreasing order. A face f is called *a face of type* $(D_1,\ldots,D_r)$ if this vector component-wise majorizes (non-strictly) the vector $D(f)$.

It is known that each plane normal map contains a face of size at most 5, called *minor*. In 1940 Lebesgue [9] proved the following fact about the structure of minor faces.

Theorem 1. *Every plane normal map has a face of one of the following types*: $(3,6,\infty)$, $(3,7,41)$, $(3,8,23)$, $(3,9,17)$, $(3,10,14)$, $(3,11,13)$, $(4,4,\infty)$, $(4,5,19)$, $(4,6,11)$, $(4,7,9)$, $(5,5,9)$, $(5,6,7)$, $(3,3,3,\infty)$, $(3,3,4,11)$, $(3,3,5,7)$, $(3,4,4,5)$, $(3,3,3,3,5)$.

☆ Translated from Discrete Analysis and Operations Research, Novosibirsk, 5, No. 4 (1998) 6–17.
* Corresponding author.
E-mail address: brdnoleg@math.nsc.ru (O.V. Borodin).
1 Supported by the Russian Foundation for Basic Research (Grants 96-01-01614 and 00-01-00916).

PII: S0166-218X(02)00292-5

The height of a face f is the maximum degree of a vertex incident with f. The bounds for the height of minor faces implied by Theorem 1 were later on improved for some special classes of graphs. Thus, Borodin [2,3,6], answering Kotzig's questions in [7,8], proved that each plane triangulation not containing faces of type $(4,4,\infty)$ has a triangle of height at most 20, and each plane graph with the minimum degree 5, a triangle of height at most 7. (As shown below, the bounds 20 and 7, are the best possible.) Avgustinovich and Borodin [1] proved that each toroidal quadrangulation, and hence each planar one, has a face of height at most 10. Hornak and Jendrol' [5] established the upper bound 23 for the height of 3-faces in the plane normal maps without faces of types $(3,5,\infty)$, $(4,4,\infty)$, $(3,3,3,\infty)$. In this paper we prove the following result.

Theorem 2. *Every plane normal map without faces of types* $(3,5,\infty)$, $(4,4,\infty)$, *and* $(3,3,3,\infty)$ *has a* 3*-face of height at most* 20, *or a* 4*-face of height at most* 11, *or else a* 5*-face of height at most* 5; *these bounds for* 3*- and* 5*-faces are best possible*.

We guess that the bound 11 above is also sharp, despite it can be improved to 10 in the class of quadrangulations (see [1]).

All conditions in Theorem 2 are necessary. Indeed, an N-bipyramid has faces of type $(4,4,N)$ only, and the dual polyhedron for the antiprism on $2N$ vertices, of type $(3,3,3,N)$ only. To justify the exclusion of the $(3,5,\infty)$-faces, we refer to the construction in [5] which has no faces of type $(4,4,\infty)$ or $(3,3,3,\infty)$ and in which each face has height at least 30.

2. Proof of the main result

The unimprovability of the bound for 3-faces follows from the graph that is obtained from the icosahedron by twice putting a new vertex in each face and joining it with the boundary vertices of the face. In the resulting graph each face has type $(3,6,20)$ or $(3,20,20)$.

The unimprovability of the bound for 5-faces follows from the graph dual to the $(3,3,3,3,5)$ Archimedean solid (see [4]).

Let G be a counterexample to Theorem 2, so that each its 3-face has a vertex v such that $d(v)\geqslant 21$, each 4-face has a vertex v with $d(v)\geqslant 12$, and each 5-face, a vertex v with $d(v)\geqslant 6$.

Euler's formula $|V|-|E|+|F|=2$ for G can be rewritten as $(2|E|-6|V|)+(4|E|-6|F|)=-12$. It follows that

$$\sum_{v\in V}(d(v)-6)+\sum_{f\in F}(2r(f)-6)=-12. \tag{1}$$

The charge of a vertex v is defined to be $d(v)-6$, and $2r(f)-6$ is *the charge of a face* f. We want to reallocate these charges, not changing their sum, so that each new charge becomes non-negative. This contradiction will complete the proof of the theorem.

First a few definitions. A vertex v in G is said to be an *m-vertex* if $3 \leqslant d(v) \leqslant 5$, an *$N$-vertex* if $6 \leqslant d(v) \leqslant 11$, an *$M$-vertex* if $12 \leqslant d(v) \leqslant 20$, and a *$B$-vertex* if $d(v) \geqslant 21$. An edge is said to be *weak* or *semiweak* if it is incident with two 3-faces or at least one 3-face, respectively.

In what follows, we shall sometimes point out the type of a face, for example, as $f = (4, B, N)$. A face of type $(3, 3, s, t)$ such that $4 \leqslant s \leqslant 5$ is called *special*.

Remark 3. We may assume that G does not contain paths $(\dots v_1, v_2, v_3, \dots)$ such that $d(v_1) \geqslant 5$, v_2 is an m-vertex, v_3 is a B-vertex, and v_1 has no common edge with v_3. Indeed, adding such an edge yields another counterexample to Theorem 2 on the same vertices. We also assume that G has no face of type $(5, B, B)$, because inserting a vertex in such a face does not violate the property of our graph to be a counterexample.

Now we specify the rules of reallocating the charges on G so that the new charge of each vertex and faces becomes non-negative, which will result in a contradiction with (1).

R1. Each face f of size at least 5 gives each incident m-vertex charge 1, with the following two exceptions:

if f has an m-vertex v lying in the boundary of f between two non-m-vertices, then v receives charge 2;

if the boundary of f contains a fragment of the type $(\dots d(v), 3, 4, 3, d(u) \dots)$, where v and u are not m-vertices, then f transmits two units of charge to the 4-vertex here.

R2. Each 4-face f transmits:

(a) 1 to every incident 3-vertex;
(b) 1 to every incident m-vertex if there are at most two such vertices in the boundary of f;
(c) if there are three m-vertices then
(c1) 0 is given to a vertex of degree 4 or 5 if f is special;
(c2) $\frac{1}{2}$ is given to a vertex of degree 4 or 5 if f is incident with only one 3-vertex, and 1 is given to a 5-vertex surrounded by two 4- or 5-vertices;
(d) 2 is given to an m-vertex if it is unique.

R3. Every M-vertex gives every incident face f charge $\frac{1}{2}$, and this charge is split as follows:

(a) if f is special then this charge is entirely transmitted to the incident 4- or 5-vertex;
(b) if f is 3-face then the charge is transmitted to the m-vertex; otherwise it is divided evenly among the m-vertices lying in this face, if any.

R4. Each B-vertex gives every adjacent vertex v of degree 3 or 4 the following charge:

(a) if $d(v) = 3$ then
(a1) along a weak edge: $\frac{3}{2}$ if v is incident with 3-faces only and is adjacent to an N-vertex;

1 if v is incident with 3-faces only but is not adjacent to an N-vertex;

1 if one of the incident faces has size at least 5, or if one of the incident faces has size 4 and the vertex lying in this face opposite to the B-vertex is not an m-vertex;

$\frac{5}{4}$ if one of the incident faces has size 4 and the vertex lying in this face opposite to the B-vertex is an m-vertex;

(a2) along a semiweak edge (observe that due to Remark 3 v can be incident in this case with only one 3-face):

1 if the third vertex in the 3-face is an N-vertex, and $\frac{1}{2}$ otherwise;

(b) if $d(v)=4$ then it receives 1 along a weak edge and $\frac{1}{2}$ along semiweak.

R5. If a B-vertex u belongs to a special face f then u transmits $\frac{1}{2}$ to the adjacent 4- or 5-vertex, except for the following cases when no charge is transferred:

(a) the B-vertex is opposite to a 4- or 5-vertex in the special face f and f has two $(3,B)$-edges in common with faces of type $(3,N,B)$;
(b) the B-vertex is opposite to a 4- or 5-vertex in f, and f has a $(3,B)$-edge in common with a face of type $(3,N,B)$ and another $(3,B)$-edge in common with a face of type $(3,B,B)$;
(c) the B-vertex is opposite to a 4-vertex in the special face f, and f has a $(3,B)$-edges in common with a face of type $(3,N,B)$ and a $(4,B)$-edge in common with a face of a type $(4,B,B)$.

R6. Suppose that B- or M-vertex u lies in a special face $f_1=(3,3,u,v)$ such that v has degree 4 and is not adjacent to u. Also suppose f_1 has common edges $(3,u)$ with non-triangular faces f_2 and f_3 each of which contains a fragment $(\ldots d(u),3,3,d(s)\ldots)$, where s is not an m-vertex. Then we declare an exception from the general rules above. Namely, if u is a B-vertex then u transmits to v charge 1 rather than $\frac{1}{2}$. If u is an M-vertex, then the charge $\frac{1}{2}$ that the vertex u transmits on f_2 and f_3 by rule R3 is divided evenly between the adjacent faces containing u. (Thus f_1 will receive an additional charge $\frac{1}{2}$ from u that goes further to v, i.e., the total charge transferred by u to v is again equal to 1.)

By $M(x)$ denote the charge obtained as a result of applying rules R1–R6. The rest of our proof consists in proving that $M(x)\geqslant 0$ whenever $x\in V\cup F$.

First consider a face f. If $r(f)\geqslant 6$ then f can give as much as 1 to each incident vertex because $2r(f)-6-r(f)\geqslant 0$. If w is not an m-vertex and it lies in the boundary of f, then w transmits $\frac{1}{2}$ to each adjacent m-vertex lying in (the boundary of) f. Suppose f has a fragment $(\ldots 3,4,3\ldots)$ bounded by non-m-vertices u and v. Then since the 3-vertices in this fragment cannot be adjacent along the cycle with two non-m-vertices (see R1), it follows that the charge $\frac{1}{2}$ which is transferred to them from each of u and v is transmitted further, on the 4-vertex. Thus the 4-vertex receives two units of charge from f in total.

Every 5-face f contains at least one non-m-vertex. Hence, f can give 1 to every (incident) m-vertex. Note that f contains at most two m-vertices adjacent to two non-m-vertices along the boundary of f. Such an m-vertex will be called *clamped*. If a clamped vertex is unique then it obtains 2, and each of the other two vertices obtains 1. If there are two clamped vertices, then each of them obtains 2 and there are no other minor vertices. If f contains a fragment $(\ldots 3,4,3\ldots)$ bounded by non-m-vertices u and v, then f does not contain any clamped 3-vertices. Therefore, according to R1, f gives 2 to the 4-vertex and 1 to each of the 3-vertices.

Every 4-face has charge 2, and it is split among the neighbor m-vertices according to R2. Due to R3, we have $M=0$ for each 3-face. Therefore, $M(f)\geqslant 0$ for each $f\in F$.

Now let v be a vertex of G. We shall distinguish 9 cases.

Case 1: $d(v)=3$. Let at v there are 3-faces only. Since G is a counterexample, it follows that v is adjacent to at least two B-vertexes. If the third vertex is an N-vertex, then by R4a1 we have $M(v)\geqslant -3+\frac{3}{2}\times 2=0$. If it is an M-vertex, then by R4a1 each B-vertex gives 1 to v, and by R3b the M-vertex gives $\frac{1}{2}$ to v through each of two 3-faces. Therefore, $M(v)=-3+1\times 2+\frac{1}{2}\times 2=0$. If v is adjacent to three B-vertices then we have $M(v)=-3+1\times 3=0$ by R4a1.

Suppose f is the only non-triangular face at v. (By Remark 3, the common vertex of the two 3-faces at v is a B-vertex, while the other two neighbors are not B-vertices.)

If $r(f)\geqslant 5$, then f gives 2 to v by rule R1, and the adjacent B-vertex gives 1 by R4a1. So $M(v)=-3+2+1=0$.

Suppose that $r(f)=4$. If the vertex which lies in f and is not adjacent to v fails to be an m-vertex, then we have $M(v)=-3+1+2=0$ by R4a1 and R2d. Otherwise v is adjacent to an M-vertex, and we have $M(v)=-3+\frac{1}{2}+\frac{1}{4}+1+\frac{5}{4}=0$ by R3b, R2a, and R4a1.

Now suppose v is incident with only one 3-face f. Then by R1 and R2a, v receives at least 1 from each non-triangular face at v. If none of the other two vertices in f is an N-vertex, then by R4a2 each of them transmits $\frac{1}{2}$ to v either along an edge (B-vertex) or through a 3-face (M-vertex). Otherwise, the B-vertex gives 1 by R4a2. This results in $M(v)=0$.

If there are no 3-faces at v, then each incident face gives 1 to v due to R1 and R2a, whence $M(v)=0$.

In the analysis of 4- and 5-vertices, we shall need the following

Lemma 4 (about exceptions). *In the exceptions described in rule R5, the* 4- *or* 5-*vertex that fails to obtain transmissions through the special face compensates its negative charge at the expense of the contributions of the other neighbor vertices and faces.*

Proof. Let v be the 4- or 5-vertex which fails to get any charge from the B-vertex u in the exceptional cases described in R5.

Case A: Observe that the faces adjacent to the special face f along the edges connecting v with 3-vertices must have size at least 4 and contain a path $(\ldots d(v),3,N\ldots)$. It follows by R1 and R2b that each such face gives v at least 1, so that $M(v)\geqslant -2+2\times 1=0$.

Case B: Consider a subpath $(\dots u_1, v_1, v, v_2, u_2 \dots)$, where all vertices except for v are adjacent to u, u_1 is an N-vertex, u_2 is a B-vertex, and v_1 and v_2 are 3-vertices incident with f. By f_i denote the face containing a fragment $(\dots u_i, v_i, v \dots)$, where $i = 1, 2$. Clearly, f_1, as well as in Case A, brings v at least 1. We can assume that $d(v) = 4$. If $r(f_2) = 5$ then f_2 also gives 1 to v by R1, whence $M(v) = 0$. Suppose $r(f_2) = 4$ (obviously, $r(f_2) \neq 3$). By f_3 denote the fourth face at v. Consider two subcases.

Suppose f_2 also belongs to the exceptions described in Lemma 4 (only Case B is possible). It means (see R5) that f_3 contains a fragment $(\dots d(v), 3, N \dots)$, and then f_3 gives v at least 1 by R1 and R2b, whence $M(v) \geqslant 0$.

Now suppose f_2 does not belong to the exceptions. Then it gives v at least $\frac{1}{2}$. Furthermore, if f_3 also does not belong to the exceptions, then f_3 gives v at least $\frac{1}{2}$, so that $M(v) \geqslant 0$. Otherwise, i.e., when f_3 is exceptional, f_1 must contain a fragment of type $(\dots N, 3, d(v), 3, d(s) \dots)$, where s is an N- or a B-vertex. Thus $r(f_3) \geqslant 5$; hence, according to R1 (observe that the 4-vertex v is clamped in the path by $(3, \geqslant 6)$-pairs), f_1 gives v charge 2 rather than 1. It follows that $M(v) = -2 + 2 = 0$.

Case C: Consider a path $(\dots u_1, v, v_1, v_2, u_2 \dots)$ whose all vertices, except for v_1, are adjacent to u, u_1 is a B-vertex, u_2 is an N-vertex, and v_1 and v_2 are 3-vertexes. By f_1 denote the face (u_1, v, u), by f_2 the face adjacent to f_1 along the edge (u_1, v), and by f_3 that adjacent to f along the edge (v, v_1).

Note that v gets at least $\frac{1}{2}$ from the vertices u and u_1 due to R4b. It is not hard to see that $r(f_3) \geqslant 4$.

If f_3 belongs to exceptions (only Case C is possible), then f_2 is a 3-face containing two B-vertexes, one of which is u_1. It follows from R4b that v gets 1 from u_1 and $\frac{1}{2}$ from the other B-vertex. Thus $M(v) = 0$.

Suppose f_3 does not belong to exceptions, i.e., v gets at least $\frac{1}{2}$ from f_3. If $r(f_2) = 3$ then by R4b u_1 gives v charge 1 rather than $\frac{1}{2}$, so that $M(v) = -2 + 1 + \frac{1}{2} \times 2 = 0$. Suppose that $r(f_2) \geqslant 4$. If f_2 is not a 4-face then it gives 1 to v, and we have $M(v) \geqslant -2 + 1 + \frac{1}{2} \times 2 = 0$. Assume $r(f_2) = 4$. If f_2 does not belong to exceptions then it gives v at least $\frac{1}{2}$, whence $M(v) \geqslant -2 + \frac{1}{2} \times 4 = 0$. If f_2 also belongs to exceptions, i.e., is special and does not transmit to v any positive charge (Case C is the only possible), then f_3 must be special. Furthermore, due to R6, the B- or M-vertex incident with f_3 now gives v charge 1 rather than $\frac{1}{2}$. Thus $M(v) = -2 + 1 + \frac{1}{2} \times 2 = 0$. This completes the proof of Lemma 4.

Due to Lemma 4, during the proof that $M \geqslant 0$ for 4- and 5-vertices we may assume that no exceptional situations happen in their neighborhood. Thus, every 4- or 5-vertex actually receives at least $\frac{1}{2}$ from each (incident) non-triangular face.

Case 2: $d(v) = 4$. First suppose that v is incident with 3-faces only. As G is a counterexample, v is adjacent to two opposite B-vertices. It then follows from R4b1 that $M(v) \geqslant -2 + 2 \times 1 = 0$. Now suppose that v is incident with only one non-triangular face. Note that any non-triangular face gives v at least $\frac{1}{2}$. Let v be adjacent to the vertices v_1, v_2, v_3, and v_4 in a cyclic order, and let the incident 3-faces be (v_4, v, v_1), (v_1, v, v_2), and (v_2, v, v_3). It means that either v_1 or v_2 is a B-vertex, and by R4b it gives 1 to v. Let it be v_1. Then either v_2 or v_3 is also a B-vertex, and thus transmits at least $\frac{1}{2}$ on v according to R4b. Furthermore, the non-triangular face gives v receives at least $\frac{1}{2}$, whence $M(v) \geqslant -2+1+\frac{1}{2}\times 2=0$.

Next suppose v is incident with precisely two 3-faces. First assume that these faces are not adjacent to each other. Then each of them contains a B-vertex, which by R4b gives v charge $\frac{1}{2}$. Furthermore, v receives at least $\frac{1}{2}$ from each of the two non-triangular faces. This implies $M(v) \geqslant -2+\frac{1}{2}\times 4=0$.

Now if these two 3-faces are consecutive, then they have either one common B-vertex, which gives 1 to v according to R4b, or two B-vertices lying in different faces, each of which gives v charge $\frac{1}{2}$ due to R4b. Furthermore, there are two non-triangular faces at v, each of which gives v at least $\frac{1}{2}$, whence $M(v) \geqslant -2+1+\frac{1}{2}\times 2=0$.

Suppose that v is incident with only one 3-face. Then this face contains a B-vertex, which gives v charge $\frac{1}{2}$ by R4b. Since each non-triangular face gives v at least $\frac{1}{2}$, we have $M(v) \geqslant -2+\frac{1}{2}\times 4=0$.

Finally, if v is not incident with 3-faces, then v receives at least $\frac{1}{2}$ from each incident face, whence $M(v) \geqslant -2+\frac{1}{2}\times 4=0$. This completes the proof for the case of 4-vertices.

Case 3: $d(v)=5$. If v is incident with a face f of size at least 5, then v receives 1 from f due to R1, i.e., $M(v) \geqslant -1+1=0$. By Remark 3, the neighborhood of v cannot consist of 3-faces only because otherwise due to the parity reasons there would exist a face of type $(5,B,B)$. If v is incident with at least two 4-faces, then each of them gives v at least $\frac{1}{2}$, whence $M(v) \geqslant -1+\frac{1}{2}\times 2=0$.

So let us assume that v is incident with only one 4-face, f. If both vertices adjacent to v and incident with f are B-vertices, then v gets at least 1 from f by R2b, so that $M(v)=-1+1=0$. If one of them is a B-vertex but the other is not, then v is incident with a face of the type $(5,B,B)$, contrary to Remark 3.

Suppose that no B-vertex is adjacent to v and incident with f. If both of such vertices are 4- or 5-vertices (clearly, none of them can have degree 3), then v obtains 1 from f by R2c2, i.e., $M(v) \geqslant -1+1=0$.

If at least one of these two vertices is an N-vertex and the other fails to be an M-vertex, then v receive at least 1 from f according to R2b, which implies $M(v) \geqslant -1+1=0$. Finally, if at least one of these vertices is an M-vertex, then v receives at least $\frac{1}{2}$ from f due to R2c2 and $\frac{1}{2}$ more from the M-vertex through the 3-face due to R3b. Thus $M(v)=-1+\frac{1}{2}\times 2=0$. The analysis of 5-vertices is completed.

Case 4: $6 \leqslant d(v) \leqslant 11$. Since the initial charges of such vertices are non-negative and no negative charges come to them according to R1–R6, we have $M(v) \geqslant 0$.

Case 5: $12 \leqslant d(v) \leqslant 20$. Now R3 implies that $M(v)=d(v)-6-d(v)\times\frac{1}{2}=(d(v)-12)/2 \geqslant 0$.

Case 6: $d(v) \geqslant 21$.

We now produce the following averaging of charges sent by v to adjacent vertices that makes it possible to give an upper bound for the total transmission of charge from v. The edges along which a charge leaves v will be called *conductors*. If a transmission should be done by v to the opposite vertex through a special face f, then we instead send a half of this charge along each of the two edges incident with v and f. The edges that are not conductors will be called *zero edges*. From each conductor, we transfer charge $\frac{3}{8}$ to the neighbor zero edge. As a result of this procedure, v will send at most

$\frac{3}{4}$ along each incident edge, and we have $M(v) \geqslant d(v) - 6 - d(v) \times \frac{3}{4} = (d(v) - 24)/4$. Therefore, we have already proved that $M(v) \geqslant 0$ whenever $d(v) \geqslant 24$.

Note that under such an averaging for $d(v) = 21$, 22, and 23 there arises a deficiency at v of $\frac{3}{4}$, $\frac{1}{2}$ and $\frac{1}{4}$, respectively. Thus, it remains to find some edges at v which take away a charge $x < \frac{3}{4}$, so that the reserves $\frac{3}{4} - x$ of such edges sum up to an amount at least as large as the above-mentioned deficiencies at v. The following two facts are easily seen.

(a) If there are two consecutive zero edges at v, then v already has a reserve of $\frac{3}{4}$ (because each of these edges remains with $\frac{3}{8}$).

(b) If v is incident with a non-triangular face f, then f creates a reserve of $\frac{1}{4}$ at v.

Indeed, suppose that both faces adjacent to f and incident with v have size 3. If $r(f) \geqslant 4$ and f is not special, then each of the two edges incident with f and v takes away at most 1 from v. It means that every such edge creates a reserve of at least $\frac{1}{8}$, i.e., v already has a reserve of $\frac{1}{4}$. Next suppose f is a special face; then v gives the vertices incident with f a total of at most 2. (Of course, we take into account the exceptions described in rule R5.) This results in a reserve of at least $\frac{1}{4}$ in this case too.

Now suppose that at least one of the faces adjacent to f at v is not triangular. Then regardless of whether f is special or not, f creates a reserve of at least $\frac{1}{4}$ at v since the total donation of v to the vertices incident with f is at most $\frac{3}{2}$.

Case 7: $d(v) = 21$. If v is incident with at least three non-triangular faces, then v has a desired reserve of $\frac{1}{4} \times 3 = \frac{3}{4}$. Suppose v is incident with precisely two non-triangular faces. Then due to the parity reasons, either one of these two faces has a zero edge at v, and the reserve on v is already at least $\frac{1}{4} + \frac{1}{2} = \frac{3}{4}$, or there are two consecutive zero edges at v.

Now suppose that v is incident with only one non-triangular face f. Then since $d(v)$ is not divisible by 4, it follows that the edges incident with f and v must go to a B-, or N-, or M-vertex, say s.

If f is not special then one of these edges transfers 1, and the other, $\frac{1}{2}$; it follows that v has a reserve of at least $\frac{3}{4}$.

Assume that f is special. Note that then s is adjacent to a 3-vertex $v_1 \in f$ such that v_1 is adjacent with v. (Otherwise, i.e., if s is adjacent with a 4- or 5-vertex, then f already creates a reserve $\frac{3}{4}$ rather than $\frac{1}{4}$.) If s is an N-vertex, then due to R5, exceptions (b) and (c), v does not transfer $\frac{1}{2}$ through the special face, which already creates a reserve of $\frac{3}{4}$. If s is an M-vertex, then due to R3b the vertex s gives v_1 an additional $\frac{1}{2}$, so that the total reserve becomes $\frac{3}{4}$.

Finally, if v is incident with 3-faces only, then the oddness of $d(v)$ implies the presence of two consecutive zero edges.

Case 8: $d(v) = 22$. If v is incident with at least two non-triangular faces, then they create the reserve of $\frac{1}{2}$ that we need. Suppose there is precisely one non-triangular face at v. Then it follows from the parity of $d(v)$ that either one of edges of this face at v

is zero, and this face alone creates a reserve of at least $\frac{1}{2}$, or there are two consecutive zero edges.

Finally, suppose that v is incident with 3-faces only. Since $d(v)$ is not divisible by 4, it follows that v is adjacent to a 3-vertex u such that either the two common neighbors w_1, w_2 of v and u are both B-vertices, or none of them is a B-vertex. (Note that none of w_1, w_2 can be an m-vertex.) In the first case, u receives 1 from v due to by R4a1, which creates a reserve $\frac{1}{2}$. Now consider the second case. If both w_1 and w_2 are N-vertices, then u receives 1 from v due to R4a1 again. If at least one of w_1 and w_2 is an M-vertex, then v is incident with two edges each taking away at most $\frac{5}{4}$ from v, which creates a reserve of $\frac{1}{4} \times 2 = \frac{1}{2}$.

Case 9: $d(v) = 23$. If v is incident with at least one non-triangular face, then it already creates a reserve $\frac{1}{4}$. Suppose otherwise; then the oddness of $d(v)$ implies the presence of two consecutive zero edges at v.

Thus, $M(v) \geqslant 0$ for all $v \in V$. This contradiction completes the proof of Theorem 2.

References

[1] S.V. Avgustinovich, O.V. Borodin, Neighborhoods of edges in normal maps, Discrete Anal. 2 (3) (1995) 3–9 (in Russian).

[2] O.V. Borodin, Solution of Kotzig's and Grunbaum's problems about the separability of cycle in plane graphs, Matem. Zametki 46 (5) (1989) 10–12 (in Russian).

[3] O.V. Borodin, Maximum weight of a face in plane triangulations, Matem. Zametki 51 (1) (1992) 16–19 (in Russian).

[4] O.V. Borodin, Triangulated 3-polytopes without faces of low weight, Discrete Math. 186 (1998) 281–285.

[5] M. Horňák, S. Jendrol', Unavoidable sets of face types for planar maps, Discuss. Math. Graph Theory 16 (2) (1996) 207–218. (Polish journal)

[6] A.B. Ivanov, Semiregular polytopes, in: Mathematical Encyclopedia, Moscow, Soviet Encyclopedia, Vol. 53, 1984, p. 463.

[7] A. Kotzig, On the theory of Eulerian polyhedra, Mat. -Fyz. Casopis 13 (1) (1963) 20–34.

[8] A. Kotzig, Extremal polyhedral graphs, Proceedings of the Second International Conference on Combinatorial Mathematics, New York Academic Sciences, New York, Ann. New York Acad. Sci. 319 (1978) 569–570.

[9] H. Lebesgue, Quelques conséquences simples de la formule d'Euler, J. Math. Pures Appl. 19 (1940) 27– 43.